$\therefore$ If a sequence is bounded below, it has infinitely many lower bounds.

Of all the lower bounds of the sequence, if K is the greatest, then K is called the greatest lower bound (g.l.u.) of the sequence.

Limit of a Sequence:-

Let $\{a_n\}$ be a sequence and $l \in R$. The real number l is said to be the limit of the sequence $\{a_n\}$ if to each $\varepsilon > 0$, $\exists$ $m \in N$ (m depending on ε) such that $|a_n - l| < \varepsilon$ $\forall$ $n \geq m$.

If l is the limit of $\{a_n\}$, then we write,

$a_n \to l$ as $n \to \infty$ or $\lim\limits_{n \to \infty} a_n = l$.

Note:- $|a_n - l| < \varepsilon$ $\forall$ $n \geq m$

$\Rightarrow$ $l - \varepsilon < a_n < l + \varepsilon$ $\forall$ $n \geq m$.

$\Rightarrow$ $a_n \in (l - \varepsilon, l + \varepsilon)$ $\forall$ $n \geq m$.

Convergent Sequence:-

If $\lim\limits_{n \to \infty} a_n = l$, then we say that the sequence $\{a_n\}$ converges to l.

Equivalently, a sequence $\{a_n\}$ is said to converge to a real number l if given $\varepsilon > 0$, however small, $\exists$ a positive integer m (depending on ε) such that

$|a_n - l| < \varepsilon$ $\forall$ $n \geq m$.

The real number l is called the limit of the sequence $\{a_n\}$.

Monotonic Sequences

(i) A sequence $\{a_n\}$ is said to be monotonically increasing if $a_{n+1} \geq a_n \;\forall\; n \in N$.

i.e. if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$

ii) A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_{n+1} \leq a_n \;\forall\; n \in N$.

i.e. if. $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$

iii) A sequence $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Note :- i.) A sequence $\{a_n\}$ is said to be strictly monotonically increasing if $a_{n+1} > a_n \;\forall\; n \in N$.

ii) A sequence $\{a_n\}$ is said to be strictly monotonically decreasing if $a_{n+1} < a_n \;\forall\; n \in N$.

* Note Properties of Least Upper Bound and Greatest lower Bound!

i)

1) It is an upper bound of the sequence. $\Rightarrow a_n \leq K \;\forall\; n \in N$.

2). Given $\varepsilon > 0$, $K - \varepsilon < K$

Since K is the least upper bound, $K - \varepsilon$ is not even an upper bound.

$\Rightarrow \exists$ at least one +ve integer m such that

$$a_m \nleq K - \varepsilon \Rightarrow a_m > K - \varepsilon.$$

ii)

1) It is a lower bound of the sequence. $\Rightarrow k \leq a_n \;\forall\; n \in N$.

2). Given $\varepsilon > 0$, $k + \varepsilon > k$.

Since k is the greatest lower bound, $k + \varepsilon$ is not even a lower bound.

$\Rightarrow \exists$ at least one +ve integer m s.t. $k + \varepsilon \nleq a_m \Rightarrow k + \varepsilon > a_m$.

Theorem 7.

Every convergent sequence has a unique limit.

Proof :- If possible, let a sequence $\{a_n\}$ converge to two distinct real numbers l and l'.

Let $\varepsilon = \frac{1}{2}|l - l'|$. Since $l \neq l'$, $|l - l'| > 0$ so that $\varepsilon > 0$.

Now the sequence $\{a_n\}$ converges to l.

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a positive integer m_1 such that
$$|a_n - l| < \frac{\varepsilon}{2} \quad \forall\, n \geq m_1$$

Also the sequence $\{a_n\}$ converges to l'.

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a positive integer m_2 such that
$$|a_n - l'| < \frac{\varepsilon}{2} \quad \forall\, n \geq m_2.$$

Let $m = \max.\{m_1, m_2\}$.

Then $|a_n - l| < \frac{\varepsilon}{2}$.

and $|a_n - l'| < \frac{\varepsilon}{2} \quad \forall\, n \geq m. \longrightarrow (1)$

Now
$$\begin{aligned}
|l - l'| &= |(l - a_n) + (a_n - l')| \\
&\leq |l - a_n| + |a_n - l'| \\
&= |a_n - l| + |a_n - l'| \quad [\because |-x| = |x|]. \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq m \quad [\text{using } (1)].
\end{aligned}$$

$\therefore \quad |l - l'| < \varepsilon \quad \forall\, n \geq m.$

which contradicts the assumption that
$$\varepsilon = \frac{1}{2}|l - l'|.$$

$\Rightarrow$ our assumption is wrong.

Hence $l = l'$.

\#

Theorem (2).

Every convergent sequence is bounded.

Pf. Let $\{a_n\}$ be a convergent sequence, converging to l.

For $\varepsilon = 1$, there exists a positive integer m such that $|a_n - l| < 1 \quad \forall\ n \geq m$.

$\Rightarrow \quad l-1 < a_n < l+1 \quad \forall\ n \geq m$.

Let $k = \min.(a_1, a_2, \ldots, a_{m-1}, l-1)$

and $K = \max.(a_1, a_2, \ldots, a_{m-1}, l+1)$.

Then $\quad k \leq a_n \leq K \quad \forall\ n \in N$.

$\Rightarrow$ The sequence $\{a_n\}$ is a bounded sequence.

Note.1

The converse of the above theorem is not true i.e. a bounded sequence is not necessarily convergent.

For example,

Consider the sequence $\{a_n\}$ defined by $a_n = (-1)^n$. We have $\{a_n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, 1 \ldots\}$ Here $-1 \leq a_n \leq 1 \ \forall\ n \in N$

Clearly the sequence is bounded, -1 is its g.l.b. (greatest lower bound) and 1 is its l.u.b. (least upper bound), but it is not convergent.

Note-2 If a sequence is not bounded, it cannot be convergent.

For example, the sequence $\{n^2\}$ is not convergent because it is unbounded.