Convergence and Divergence of Infinite Series

Defn:

Series: An expression of the form $u_1 + u_2 + u_3 + \ldots$ in which the successive terms are formed according to some definite law is called a series.

Finite Series: A series in which there is a limited no. of terms is called a finite series.

Infinite Series: A series in which there is no limit to the no. of terms, or one in which every term is followed by another is called an infinite series.

Convergent, Divergent and oscillatory Series:-

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent if $\lim_{n \to \infty} S_n$ exists finite, where $S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \ldots + u_n$.

If $\lim_{n \to \infty} S_n$ is infinite, then the series is said to be Divergent.

A series which is neither convergent nor divergent is called an oscillatory series.

Comparison tests for +ve term series:-

If $a \le \frac{u_n}{v_n} \le b$ for all values of $n > m$ where a and b are +ve constants or if $\lim_{n \to \infty} \frac{u_n}{v_n} = k \neq 0$; the two series $\sum u_n$ and $\sum v_n$ converge or diverge simultaneously.

Further if $\lim \frac{u_n}{v_n} = 0$ and $\sum v_n$ converges then $\sum u_n$ also converges and if $\lim \frac{u_n}{v_n} \to \infty$ and $\sum v_n$ diverges then also $\sum u_n$ diverges.

S. The infinite series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots \infty$ is convergent if $p > 1$ and divergent if $p = 1$ or < 1.

Case-I When $p > 1$

We have $\frac{1}{1^p} = 1$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p}$$

$p > 1$
$3^p > 2^p$
$\frac{1}{3^p} < \frac{1}{2^p}$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} < \frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} = \frac{4}{4^p} \left(\frac{2}{2^p}\right)^2$$

$\therefore$ $S < 1 + \frac{2}{2^p} + \left(\frac{2}{2^p}\right)^2 + \dots$

which is a geometric series with common ratio $\frac{2}{2^p}$

Since $p > 1$, $2^p > 2$

$\Rightarrow \frac{1}{2^p} < \frac{1}{2}$

$\Rightarrow \frac{2}{2^p} < 1$

Hence $1 + \frac{2}{2^p} + \left(\frac{2}{2^p}\right)^2 + \dots$ is convergent and it converse to $\frac{1}{1-\frac{2}{2^p}}$

In G.P. series
$1 + r + r^2 + \dots + \infty$
$\therefore S_n = \frac{a}{1-r}$

$\therefore$ The series $\frac{1}{1^p} + \frac{1}{2^p} + \dots$ is convergent for $p > 1$.

Case-II: when $p = 1$

$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

$1 = 1$

$\frac{1}{2} = \frac{1}{2}$

$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

$\therefore S > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$ which is divergent.

$\therefore$ S is divergent.

Case-III when $p < 1$

$S = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$

$\frac{1}{2^p} > \frac{1}{2}$

$\frac{1}{3^p} + \frac{1}{4^p} > \frac{1}{4^p} + \frac{1}{4^p} = \frac{2}{4^p} > \frac{2}{4} = \frac{1}{2}$

$p < 1$
$2^p < 2$
$\frac{1}{2^p} > \frac{1}{2}$

$\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} > \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} > \frac{4}{8^p} > \frac{4}{8} = \frac{1}{2}$

$\therefore$ $S > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$ which is divergent.

Hence S is also divergent.

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p = 1$ and $p < 1$.

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Exercise – V(A) (Book – M. Ray).

Determine whether the following Series are Convergent or divergent :-

1. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots\cdots$

Soln :- Here $u_n=\frac{1}{(2n-1)\,2n}$

Let $v_n=\frac{1}{n^2}$

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{1}{2n(2n-1)}}{\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}\frac{n}{2(2n-1)}=\lim_{n\to\infty}\frac{1}{2\left(2-\frac{1}{n}\right)}$$

$$=\frac{1}{4}\neq 0.$$

Now $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is convergent [$\because p=2>1$]

$\therefore \sum_{n=1}^{\infty}u_n$ is also convergent. #

2. $1-\frac{1}{a+1}+\frac{1}{1+2a}-\frac{1}{1+3a}+\cdots$ if $a>1$.

Soln :- Here $u_n=\frac{1}{1+(n-1)a}$

$v_n=\frac{1}{n}$.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{1}{1+(n-1)a}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{1+(n-1)a}$$

$$=\lim_{n\to\infty}\frac{1}{\frac{1}{n}+\left(1-\frac{1}{n}\right)a}=\frac{1}{a}\neq 0$$

Now $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent.

$\therefore \sum_{n=1}^{\infty}u_n$ is also divergent. #

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$

Soln :- Here $u_n=(-1)^{n-1}\frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty}u_n=0$$

It is an alternating Series and each term is numerically less than the preceding term. Hence it is convergent.